14-4 Lines in Space

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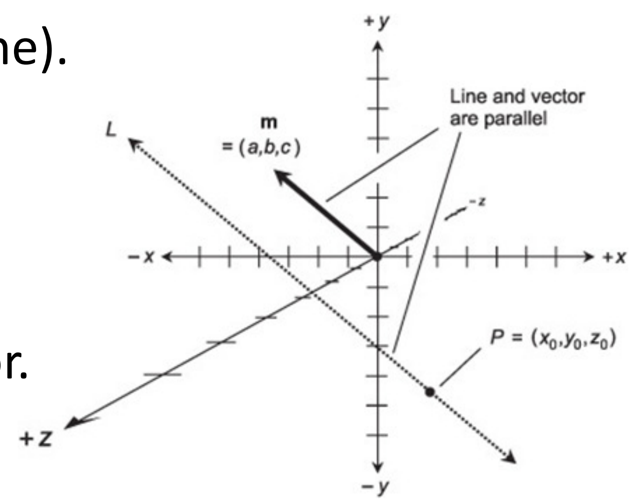
A line in space is uniquly determined by 2 points on the line (just like in the plane).

It is also determined by a

point and a direction vector.



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Let A (x_0, y_0, z_0) be a point on the line and let $\overrightarrow{AM}$ <a, b, c> represent the direction vector. The point M(x, y, z) lies on the line if $\overrightarrow{AM} = < x - x_0, y - y_0, z - z_0 >$ is a scalar multiple of <a,b,c>. So any point that satisfies the equation is on the line.

$$< x - x_0, y - y_0, z - z_0 > = t < a, b, c >$$
$$(x, y, z) - (x_0, y_0, z_0) = t < a, b, c >$$
$$(x, y, z) = (x_0, y_0, z_0) + t < a, b, c >$$

EQUATIONS OF A LINE IN SPACE

Vector Equation of a Line

$$\boxed{(x, y, z) = (x_0, y_0, z_0) + t < a, b, c >}$$

Parametric Equations of a Line

$$x - x_0 = ta$$
$$y - y_0 = tb$$
$$z - z_0 = tc$$

$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}}$$

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Ex1. Given the points A(1,3,-1) & B(2,5,0), represent the line through A and B by:

a.) A Vector Equation

b.) Parametric Equation

c.) Cartesian Equation (Symmetric)

a.) $\boxed{(x,y,z) = (1,3,-1) + t\langle 1,2,1\rangle}$

b.) $$\boxed{\begin{aligned} x &= 1+t \\ y &= 3+2t \\ z &= -1+t \end{aligned}}$$

c.) $t = x-1 \qquad t = \frac{y-3}{2} \qquad t = z+1$

$$\boxed{x-1 = \frac{y-3}{2} = z+1}$$

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Line Segments

Ex2. A line segment has endpoint A(1,3,-1) and B(2,5,0). Write the vector and parametric equations of the line segment.

$$\boxed{(x,y,z) = (1,3,-1) + t\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle \quad 0 \le t \le \sqrt{6}}$$

$$(2,5,0) = \cancel{(1,3,-1)} + t\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$-(1,3,-1) \quad \cancel{-(1,3,-1)}$$

$$(1,2,1) + t\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$\frac{\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle}{} \quad \frac{}{\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle}$$

$$\sqrt{6} = t$$

Intersecting, Parallel, and Skew Lines

Parallel Lines

Two lines are parallel if their direction vectors are scalar multiples of each other

You could also find the angle between them and it would be either 0^0 or 180^0.

You also need to check to make sure they aren't the same line by making sure they don't have any points in common.

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Intersecting, Parallel, and Skew Lines

Intersecting Lines

To determine if two lines are intersecting, solve the system of equations to find the point of intersection.

If the system doesn't have a unique solution, then the lines are not intersecting.

You have to make sure that they are not "intersecting everywhere" or in other words are the same line (coincident).

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Intersecting, Parallel, and Skew Lines

Skew Lines

If the lines are not parallel, intersecting, or coincident, then they are skew.

Ex3. Are the following lines parallel, intersecting, skew, or coincident. If they are intersecting, find the point of intersection.

1.) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ & $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$

$$-2 \times \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$$

parallel

$$\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \neq t\begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$$

2.) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 6 \\ 6 \\ 2 \end{pmatrix} = \mu \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

$$\mu = 2$$

Coincident

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3.) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} + \omega \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \varepsilon \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

$$\cancel{\begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}} + \omega \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \varepsilon \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$-\cancel{\begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}} \qquad\qquad -\begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$$

$$\omega \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -16 \end{pmatrix} + \varepsilon \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$\omega \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \varepsilon \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -16 \end{pmatrix}$$

$$\boxed{\begin{aligned} -1\omega - 2\varepsilon &= -4 \\ 2\omega + 2\varepsilon &= 0 \end{aligned}}$$

$$1\omega - 3\varepsilon = -16$$

$$\begin{aligned} -1w - \cancel{2\varepsilon} &= -4 \\ +\quad 2w + \cancel{2\varepsilon} &= 0 \\ \hline 1w &= -4 \\ w &= -4 \end{aligned}$$

$$\begin{aligned} -1(-4) - 2\varepsilon &= -4 \\ 4 - 2\varepsilon &= -4 \\ -4 \qquad &\quad -4 \\ \hline -2\varepsilon &= -8 \\ \varepsilon &= 4 \end{aligned}$$

$$1(-4) - 3(4) = -16$$

$$-4 - 12 = -16$$

intersecting

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} + \omega \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \varepsilon \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + 4 \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \begin{pmatrix} 8 \\ -8 \\ 12 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -7 \\ -2 \end{pmatrix}}$$

4.) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ & $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$$

$$\beta \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + \delta \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$$

$$\beta \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \delta \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$-\beta - 3\delta = 2$$
$$\begin{bmatrix} \beta + 4\delta = -3 \\ 2\beta - 0\delta = 2 \end{bmatrix}$$

$2\beta = 2$

$\beta = 1$

$1 + 4\delta = -3$

$\delta = -1$

$$-1 + 4 \stackrel{?}{=} 2$$
$$3 \neq 2$$

Applications to Motion

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\underbrace{\left|\vec{v}\right|}_{\text{speed}}\begin{pmatrix} \frac{d_x}{\left|\vec{d}\right|} \\ \frac{d_y}{\left|\vec{d}\right|} \\ \frac{d_z}{\left|\vec{d}\right|} \end{pmatrix}$$

→ unit vector for $\vec{v}$

$\vec{d}$ = Displacement vector

$\left|\vec{v}\right|$ = Speed

Ex4. A damaged yellow submarine rests on the bottom of the ocean at the coordinates (150,270,-1/2). A rescue submarine is located at the following coordinates (10,400,-1/4) where all distances are given in miles. The top speed of the rescue sub is 30 mph. Find:

a.) The unit vector describing the submarines direction of motion

$\vec{v} = \langle 140, -130, \frac{-1}{4} \rangle$

$\sqrt{140^2 + (-130)^2 + \left(\frac{-1}{4}\right)^2} = \sqrt{36500.0625}$

$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{140}{\sqrt{36500.0625}}, \frac{-130}{\sqrt{36500.0625}}, \frac{-\frac{1}{4}}{\sqrt{36500.0625}} \right\rangle$

b.) The vector equation modelling the submarines path.

$$(x, y, z) = (10, 400, -\tfrac{1}{4}) + 30t \left\langle \frac{140}{\sqrt{36500\tfrac{1}{16}}}, \frac{-130}{\sqrt{36500\tfrac{1}{16}}}, \frac{-\tfrac{1}{4}}{\sqrt{36500\tfrac{1}{16}}} \right\rangle$$

c.) Find out how long it takes for the rescue sub to reach the damaged sub?

$$\begin{pmatrix}150\\270\\-1/2\end{pmatrix}=\begin{pmatrix}10\\400\\-1/4\end{pmatrix}+30t\begin{pmatrix}140/\sqrt{36500.0625}\\-130/\sqrt{36500.0625}\\1/4/\sqrt{36500.0625}\end{pmatrix}$$

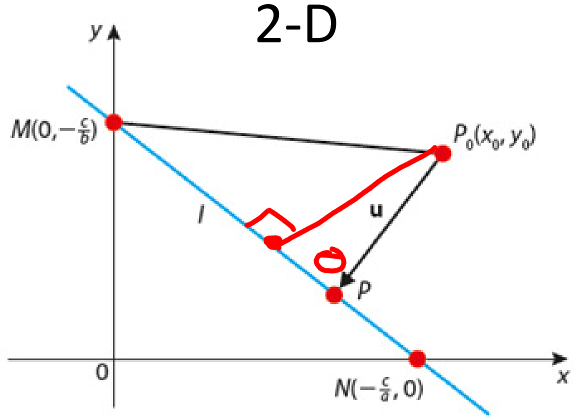
$$\overset{140}{\cancel{150}}=\cancel{10}+30t\left(140/\sqrt{36500.0625}\right)$$

$$\frac{140}{30}=t\left(140/\sqrt{36500.0625}\right)$$

$t=6.37$ hours

Distance from a Point to a Line

2-D



If the equation of the line $\boldsymbol{\ell}$ is $ax + by + c = 0$ and point $P(x_0, y_0)$ is not on a line $\boldsymbol{\ell}$, then the distance from P to $\boldsymbol{\ell}$ is given by:

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

b.) Find the distance between $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$.

$P_1(-1,1,1)$ $\quad \vec{v}_1 = \langle 2,-2,4 \rangle$

$P_2(1,-1,4)$ $\quad \vec{v}_2 = \langle 2,-1,3 \rangle$

$\overrightarrow{P_1P_2} = \langle 2,-2,3 \rangle$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 2 & -2 & 4 \\ 2 & -1 & 3 \end{vmatrix} = i\begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} - j\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} + k\begin{vmatrix} 2 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= i(-6+4) - j(6-8) + k(-2+4)$$

$$= i(-2) - j(-2) + k(2)$$

$$= -2i + 2j + 2k$$

$$|\vec{v}_1 \times \vec{v}_2| = \sqrt{(-2)^2+(2)^2+(2)^2} = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

$$d = \left| \frac{\langle 2,-2,3 \rangle \cdot \langle -2,2,2 \rangle}{2\sqrt{3}} \right| = \left| \frac{-4+-4+6}{2\sqrt{3}} \right|$$

$$= \frac{-2}{2\sqrt{3}} = \left| \frac{-1}{\sqrt{3}} \right| = \frac{\sqrt{3}}{3}$$

Ex5. A line $\boldsymbol{\ell}$ has equation x + 3y – 7 = 0 and point P has coordinates (5,-1). Find the shortest distance from P to $\boldsymbol{\ell}$.

$\ell: x+3y-7=0$

$1=a,\ 3=b,\ -7=c$

$P: (5,-1)$

$5=x_0$

$-1=y_0$

$$d=\frac{|(1)(5)+(3)(-1)+(-7)|}{\sqrt{(1)^2+(3)^2}}$$

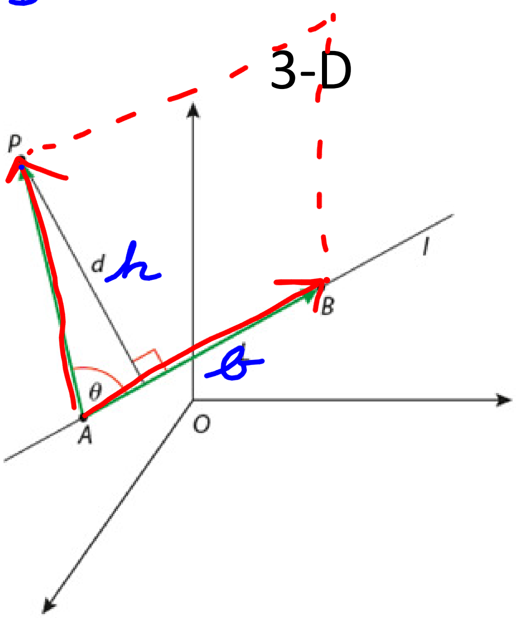
$$d=\frac{5}{\sqrt{10}}=\frac{\sqrt{10}}{2}$$

$A = b \cdot h$

$\frac{A}{b} = h$

Distance from a Point to a Line

3-D



If A is any point on the line ℓ and $\vec{L}$ is a vector parallel to ℓ, then the distance from P to ℓ is given by:

$$d = \frac{\left|\vec{L} \times \overrightarrow{AP}\right|}{\left|\vec{L}\right|}$$

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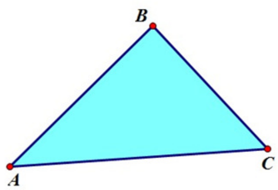
Why?



$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$\frac{\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}{\cancel{1/2}\left|\overrightarrow{AC}\right|} = h$$

$$\frac{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}{\left|\overrightarrow{AC}\right|} = h$$

$$Area = \frac{1}{2} b \cdot h$$

$$\frac{Area}{\cancel{1/2}\, b} = \frac{\cancel{1/2}\, b \cdot h}{\cancel{1/2}\, b}$$

$$\frac{Area}{\cancel{1/2}\, b} = h$$

Ex6. Consider the point P(-1,2,3) and the line $\boldsymbol{\ell}$ with parametric equations

$x = 1+2t$

$y = -4 + 3t$

$z = 3 + t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}}_{A} + t\underbrace{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}_{\vec{L}}$$

$\vec{AP}$

Find the shortest distance from P to l.

$d = \dfrac{|\vec{L} \times \vec{AP}|}{|\vec{L}|}$ ← area = bh ← divide by base $\frac{A}{b}$ = h of L

P(-1,2,3) ℓ $x = 1+2t$, $y = -4+3t$, $z = 3+t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \qquad d = \frac{\left|\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}\right|}{\sqrt{2^2+3^2+1^2}}$$

$\vec{AP}$: $\langle -2, 6, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ -2 & 6 & 0 \end{vmatrix} = i\begin{vmatrix} 3 & 1 \\ 6 & 0 \end{vmatrix} - j\begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} + k\begin{vmatrix} 2 & 3 \\ -2 & 6 \end{vmatrix}$$

$i(0-6) - j(0+2) + k(12+6)$

$-6i - 2j + 18k$

$$\frac{\sqrt{6^2+2^2+18^2}}{\sqrt{2^2+3^2+1^2}} \qquad \frac{\sqrt{364}}{\sqrt{14}} = \boxed{\sqrt{26}}$$

Distance Between 2 Skew Lines

The distance between 2 skew lines is the length of the common perpendicular between them. This is the shortest distance between these lines. Let line ℓ_1 have direction vector v_1 and contain point P_1 and let line ℓ_2 have direction vector v_2 and contain the point P_2. If lines ℓ_1 and ℓ_2 are skew, then the shortest distance between these two lines is given by:

$$d = \frac{\left| \overrightarrow{P_1P_2} \cdot \left(\overrightarrow{v_1} \times \overrightarrow{v_2}\right) \right.}{\left|\overrightarrow{v_1} \times \overrightarrow{v_2}\right|}$$

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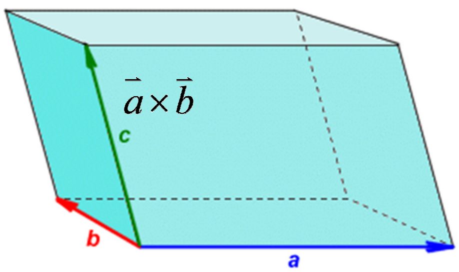
Why?



$$Volume = b \cdot l \cdot h$$

$$\frac{Volume}{b \cdot l} = \frac{b \cdot l \cdot h}{b \cdot l}$$

$$\frac{Volume}{b \cdot l} = h$$

$$Volume = \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|$$

$$Area = \left|(\vec{a} \times \vec{b})\right|$$

$$h = \frac{\left|\vec{c} \cdot (\vec{a} \times \vec{b})\right|}{\left|\vec{a} \times \vec{b}\right|}$$

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Ex7. Line ℓ_1 is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

and line ℓ_2 is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

a.) Show that ℓ_1 and ℓ_2 are skew.

$-1+2t = 1+2s$
$1+-2t = -1-s$
$1+4t = 4+3s$

$\cancel{-1}+\cancel{2t} = \cancel{1}+2s$
$+\ \cancel{1}+\cancel{-2t} = \cancel{-1}+s$
$0 = 3s \quad s = 0$

$1+4(1) = 4+3(0)$
$1+4 = 4+0$
$5 \neq 4$

$-1+2t = 1+2s$
$-1+2t = 1$
$2t = 2 \quad t = 1$

Hence ℓ_1 and ℓ_2 are skew

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